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Kinetic equation and clipping - two limits of wave turbulence theory.*

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Abstract

Different dynamics, described by kinetic equation and clipping method is shown as well as a role of approximate resonances in wave turbulence theory. Applications of clipping method are sketched for gravity-capillary and drift waves. Brief discussion of possible transition from continuous spectrum (= kinetic equation) to discrete spectrum (= clipping) is given at the end.

1 Introductory remarks

Wave kinetic equation has been developed in 60-th (see, for instance, [7], [11]) and applied for many different types evolution PDEs. Kinetic equation is approximately equivalent to the initial nonlinear PDE but has more simple form allowing direct numerical computations of each wave amplitudes in a given domain of wave spectrum. Wave kinetic equation is an averaged equation imposed on a certain set of correlation functions and it is in fact one limiting case of the quantum Bose-Einstein equation while the Boltzman kinetic equation is its other limit. Some statistical assumptions have been used in order to obtain kinetic equations and limit of its applicability then is a very complicated problem which should be solved separately for each specific

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equation [17].

Simply formulated, conditions used for obtaining of kinetic equation can be described as following:

- each wave takes part in resonant interactions,
- each wave interacts with all other waves simultaneously,
- all wave amplitudes are of the same order.

As a result, original evolution equation has been reduced to an equation of a form

$$\frac{d}{dT}A_i = \int \mathcal{G}(\vec{k}_i)\delta(\sum \vec{k}_i)d\vec{k}_i$$

which is solved in respect to each separate wave amplitude A_i. Here function $\mathcal{G}$ depends on the form of initial nonlinear PDE and notation $\delta(\sum \vec{k}_i)$ is used for delta-function which is equal to zero on the solutions of the system of equations describing exact resonances among interacting waves

$$\begin{cases} \omega(\vec{k}_1) \pm \omega(\vec{k}_2) \pm ... \pm \omega(\vec{k}_{n+1}) = 0, \\ \vec{k}_1 \pm \vec{k}_2 \pm ... \pm \vec{k}_{n+1} = 0. \end{cases} \tag{1}$$

and is non-zero otherwise. Wave kinetic equation is often written out in the form

$$\frac{d}{dT}A_i^2 = \int \mathcal{G}(\vec{k}_i)\delta(\sum \vec{k}_i)d\vec{k}_i$$

because amplitude square is proportional to the wave energy and this form allows to treat the results produced by kinetic equation in terms of energy exchange between the interacting waves.

Existence of resonances, i.e. solutions of Sys.(1), in a $(n+1)$-wave system for some specific PDE allows us to replace the PDE by kinetic equation which govern energy transfer through the spectrum. A very interesting fact is that even some classification of dispersive PDEs due to their integrability properties was constructed basing on solutions of Sys.(1). We present it below very briefly and for more details see [16].

First of all, it was proven that in case when Eq.(1.1) does not have any solutions, initial nonlinear PDE could be transformed into some linear PDE by canonic transformations. If, on the other hand, Eq.(1.1)

does have some solutions, then original PDE still has nonlinearity after canonic transformation and it can be written out as

$$\Sigma_i \frac{T_i \delta(\vec{k}_1 \pm \vec{k}_2 \pm ... \pm \vec{k}_i)}{\omega(\vec{k}_1) \pm \omega(\vec{k}_2) \pm ... \pm \omega(\vec{k}_i)}$$

where δ is a delta-function. Vertex coefficient T_{i_0} for some specific i_0 is just generalization of interaction coefficient from the system for slowly changing amplitudes in case of i_0-waves interactions. In case of zero vertex coefficient, $T_i = 0$ we fall into the class of equations having solitonic solutions and in case of non-zero vertex coefficients, as it was shown before, the kinetic equation is constructed which to some extent is equivalent to the original PDE.

Kinetic equations have been written for infinite interaction domains, i.e. for an infinite plane or an infinite channel, and were used successfully for about 25 years for description of many types of waves, mostly in cases for 3- and 4-waves interactions. The results of laboratory experiments showed that "reasonable agreement is normally obtained with theoretical results for the infinite case" [2], if the wavelength are small enough in comparison to the size of the experimental basin. The cases when wavelengths are comparable with the characteristic sizes of the experimental basin remain, as a rule, unexplained [6] and have been named "effects of finite lengths" [2]. Attempts to put some additional physically relevant terms into the kinetic equations in order to make them applicable to long-wavelength systems have failed. The author of the pioneering work in this field, O. Phillips, who obtained the first kinetic equation in [11] wrote in [12] that "new physics, new mathematics and new intuition is required" in order to understand energetic behavior of large-scale systems).

Roughly speaking, large-scale wave system is a system where wave do notice the boundaries. It means that original PDE has to be regarded with zero or periodical boundary conditions and correspondingly wave spectrum is discrete. Now Sys.(1) is a system of Diophantine or algebraic equations to be solved in integers and assumptions used for obtaining of kinetic equations in case of continuous wave spectrum together with discrete Sys.(1) have to be scrutinized closely. Why do any pair of integers (m, n) has to be part of some solution? Or, even more, to be part of infinitely many solutions? Is it really true that this algebraic system always has infinitely many solutions? Rather not. At least, one has to prove it.

It was shown [8], [9] that the study of discrete systems demands a

new approach, different from method of kinetic equation, due to their quite different properties:

- all interacting waves are partitioned into disjoint classes, there is no energy flow between these classes and it is possible to study each class independently; the number of elements of a class is not large, mostly classes consist of a few waves only;
- there are many waves which do not take part in any resonant interactions at all (for instance, in case of atmospheric planetary waves this amounts to 60 % of all waves);
- wave interaction is local in a sense that for any fixed wave its interaction domain (i.e. the one which contains all the waves which can interact with a given one) is finite and can be written out explicitly;
- number of interacting waves depend drastically on the shape of a basin (for example, on the ratio of its sides); there exist many basin where wave interactions are not possible; for some fixed dispersion ø it is possible to describe set of all such basins.

In particular, it means that discrete wave systems can not be described in terms of kinetic equation and has to be studied separately.

All the results [9] were obtained for exact resonances while in [10] it was shown that specific properties of discrete systems are still valid for some non-zero resonance width

$$0 < \Omega = \omega(\vec{k}_1) \pm \omega(\vec{k}_2) \pm ... \pm \omega(\vec{k}_1). \tag{2}$$

The existence of allowed resonance width gave rise in [10] to formulation of Clipping method and these results we briefly present in Section 2, just for completeness of exposition. which we briefly present in Section 2. Applications of Clipping method for two physical problems are demonstrated in Sections 3 and Section 4. Brief discussion is given in Section 5.

2 Approximate resonances

In order to answer the questions about resonance width, we need some estimation of Ω as a linear form on the value of the function ω taken in different points of its definition domains depends on the ω changing domain. Let us consider cases.

1. Let $\omega : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$, where $\mathbb{Z}$ denotes the set of integers and $\mathbb{Q}$ denotes the set of rational numbers. Then obviously Ω can be

represented as difference of two rational numbers a/b and c/d and we have trivial estimation

$$|\Omega| = |\frac{a}{b} - \frac{c}{d}| \geq \frac{1}{bd} > 0, \tag{3}$$

As illustrative example for this case one may take spherical Rossby waves with $\omega = -\frac{2m}{n(n+1)}$.

2. Let $\omega : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}(\alpha)$, where α is an algebraic number, i.e. it is a zero of some polynomial

$$P(x) = a_0 x^r + a_1 x^{r-1} + ... + a_r,$$

where a_i not all equal to zero. The field $\mathbb{Q}(\alpha)$ denotes the algebraic expansion of $\mathbb{Q}$, i.e., the set of numbers of the form

$$\frac{a}{b} + \frac{c}{d}\alpha,$$

with $a, b, c, d \in \mathbb{Q}$. In this case we may use generalization of the Thue-Siegel-Roth theorem [15]: If the algebraic numbers $\alpha_1, \alpha_2, ..., \alpha_s$ are linearly independent with 1 over $\mathbb{Q}$, then for any $\epsilon > 0$ we have

$$|q_1\alpha_1 + q_2\alpha_2 + ... + q_s\alpha_s - p| > cq^{-s-\epsilon},$$

for all $p, q_1, q_2, ..., q_s \in \mathbb{Z}$ with $q = \max_i |q_i|$. The constant c has to be constructed for every specific set of algebraic numbers separately. As illustrative example for this case one may take capillary waves with $\omega = (m^2 + n^2)^{3/2}$.

3. Let $\omega : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers, ω is an arbitrary real-value function of integer variables. To find frequency discrepancy Ω in the *finite* spectral domain D with some T such that

$$D = \{(m, n) : 0 < m, n \leq T < \infty\},$$

it is enough to calculate it as

$$\Omega = min_p \Omega_p,$$

where

$$\begin{aligned} \Omega_p = \omega(\vec{k}_1^p) \pm \omega(\vec{k}_2^p) \pm ... \pm \omega(\vec{k}_s^p), \\ \vec{k}_j^p = (m_j^p, n_j^p), \quad \forall j = 1, 2, ...s, \\ \omega(\vec{k}_1^p) \pm \omega(\vec{k}_2^p) \pm ... \pm \omega(\vec{k}_s^p) \neq 0 \quad \forall p, \end{aligned}$$

and p is finite because the total number of wave vectors belonging to D is finite. So defined Ω_p obviously is a non-zero number as a minimum of finite number of non-zero numbers. As illustrative example for this case one may take gravity water waves with $\omega = k \tanh \alpha k$.

Therefore, for a wide class of dispersion functions ω it is possible to find the low boundary for the frequency discrepancy Ω; in a *finite spectral domain* this boundary exists for *arbitrary* ω.

Of course, in order to apply the theoretical results of wave turbulence theory for some specific physical problem, one has also to compute values of wave amplitudes corresponding to the regime of weak turbulence. In other words, one has to characterize explicitly the nonlinearity of a system under consideration, i.e. to choose a small parameter $0 < \varepsilon << 1$ describing a weakly nonlinear regime. For instance, for planetary waves the ratio of the particle velocity to the phase velocity is often taken as the small parameter which makes it possible to obtain [10] following estimation for a wave amplitude $a(m,n)$ with a wave vector $\vec{k} = (m,n)$:

$$|a(m,n)| < \frac{6mn!2^{2n+1-m}}{5n(n+1)^{m+n+3}(n-m)(5n-m-3)}.$$

Usually it is a non-trivial task to compute small parameter ε symbolically, and also technical facilities has to be taken into account (see Section 3) in order to get some plausible results when planning some laboratory experiment.

It is important to realize that in order to observe effects predicted by wave turbulence theory, one has to compute together 1) allowed resonance width, and 2) allowed magnitudes of resonantly interacting waves.

3 Gravity-capillary waves

Now that we know exact results and are able to calculate some small but non-zero resonance width let us plan some laboratory experiment - an experimental procedure which allows laboratory experiments to be performed in order to observe how our theory works in reality. Water waves are probably the most appropriate class of wave systems for the experimental performance because their properties have been studied profoundly and the experimental facilities developed are very sophisticated indeed. We regard here

dispersion relation for gravity-capillary waves

$$\omega^2(\vec{k}) = gk + \frac{\mu}{\nu}k^3$$

where h is undisturbed depth, g is the gravitational acceleration, ν is the density, μ is the surface tension, $\vec{k}$ is the wave vector, $k = |\vec{k}|$ and in c.g.s. units g=981, $\mu = 74$ (for water "filtered of particles with nominal sizes larger than 0.2 μm and with temperature about 18° C", see [6], p.55), $\nu = 1$. The ratio μ/ν may vary considerably even for the same liquid, for instance, if the water is not filtered of particles then $\mu/\nu = 54$; this ratio may also depend on the temperature [3]. Therefore, first of all we have to choose the spectral domain of interest and then find resonantly interacting waves corresponding to the given ratio μ/ν. Let us regard spectral domain $k_x, k_y \leq 30$ which corresponds to wavelengths in the range (0.1, 3.0 cm) and find corresponding solutions of resonance conditions in the form:

$$\begin{aligned} \omega(\vec{k}_1) + \omega(\vec{k}_2) = \omega(\vec{k}_3), \\ k_{x_1} + k_{x_2} = k_{x_3}, \quad k_{y_1} + k_{y_2} = k_{y_3}, \\ k_{x_i}, k_{y_i} \in \mathbb{N}, \quad i = 1,2,3 \quad k_{x_i}, k_{y_i} \leq 30. \end{aligned}$$

Choice of frequencies to generate

Let us notice that when theoretically we have to find only *exact* solutions, in practice all the solutions with small enough frequency discrepancy have also to be considered. The ratio D of frequency discrepancy to the minimal driving frequency can be chosen as the characteristic of the discrepancy smallness and the value of D is obviously connected with experimental precision. Driving frequency can be controlled with precision of order 10^{-5} [4]; therefore we may only consider as resonant triads with small enough D, say $D = 10^{-6}$. Below we present a few solutions (three couples of numbers in square brackets denote the wave numbers of three wave vectors while three numbers in round brackets mean the three corresponding frequencies in Hz) for different liquids:

$TypeA$

$\mu/\nu = 75: \quad [1,2][9,1][10,3]; (8.7638, 40.4435, 49.2073)$

$\mu/\nu = 47: \quad [1,26][16,4][17,30]; (147.0295, 75.8317, 222.8612)$

$\mu/\nu = 27: \quad [1,10][28,6][29,16]; (30.7235, 129.5023, 160.2258)$

$\mu/\nu = 16: \quad [1,6][4,5][5,11]; (15.5681, 16.2945, 31.8626)$

We present here just a few solutions from the multitude of existing solutions in order to show that for wide variety of liquids resonant triads do exists (the value of $\mu/\nu = 75$ corresponds to clear water, 8° C; $\mu/\nu = 47$: glycerine $C_3H_5(OH)_3$, 20° C; $\mu/\nu = 27$: benzol C_6H_6, 60° C; $\mu/\nu = 16$: benzaldehyde C_6H_5CHO film on water, 20° C).

In order to demonstrate one of the most striking properties of discrete resonant systems - namely, the existence of many non-interacting waves - we have also to compute the frequencies of the the waves providing big frequency discrepancy. To find them, it is enough to solve the system of equations

$$\begin{aligned} D = max_{\vec{k}_1,\vec{k}_2,\vec{k}_3} D_{123}, \\ D_{123} \times min(\omega(\vec{k}_1), \omega(\vec{k}_2), \omega(\vec{k}_3)) = \\ = \omega(\vec{k}_1) + \omega(\vec{k}_2) - \omega(\vec{k}_3), \\ k_{x_1} + k_{x_2} = k_{x_3}, \quad k_{y_1} + k_{y_2} = k_{y_3}, \\ k_{x_i}, k_{y_i} \in \mathbb{N}, \quad i = 1,2,3, \quad k_{x_i}, k_{y_i} \leq 30. \end{aligned}$$

As the solutions of this system we obtain the triads of waves with the maximal possible ration of frequency discrepancy to the minimal driving frequency in the given spectral domain $k_{x_i}, k_{y_i} \leq 30$ and for given μ/ν (it is enough indeed to find triads with discrepancies larger than experimental precision, i.e. to replace its first equation by $D > 0.1$. Below a few triads with large discrepancies are presented:

$TypeB$

$\mu/\nu = 75:$ $[11,15][14,15][25,30]; (112.6460, 130.0788, 337.7987)$
$\mu/\nu = 47:$ $[14,14][15,16][29,30]; (98.6504, 114.4728, 295.8396)$
$\mu/\nu = 27:$ $[4,4][26,26][30,30]; (16.2595, 186.8502, 230.8321)$
$\mu/\nu = 16:$ $[5,5][25,25][30,30]; (17.8606, 137.0759, 178.8991)$

Thus now we have two types of triads: type A (exact resonant triads) and type B (triads with big discrepancies). Excitation of the frequencies corresponding to A-triads provides the possibility to observe standard periodic energy exchange within the triad. Excitation of the frequencies corresponding to some B-triad will not provide any periodic motion for similar initial conditions (i.e. for the same experimental facilities, the same liquid and the same magnitudes of initial wave amplitudes).

Choice of initial amplitudes

Now we have found the frequencies of waves to generate but we still have to choose initial values for the wave amplitudes. The resonant interaction theory deals with weak nonlinearity, i.e. some small parameter $0 < \epsilon << 0$ has to be chosen in order to estimate how big the initial amplitudes are allowed to be: they have to be big enough in order to eliminate the linear wave propagation but at the same time the amplitudes have to be not too large in order to escape turbulence.

There exist different ways of choosing ε, the choice is defined by the specifics of the wave system. For instance, as it was mentioned above, for spherical planetary waves ε is chosen as a ratio of the particle velocity to the phase velocity while for water waves usually $\varepsilon = ak$, where a is amplitude of a wave and $k = |\vec{k}|$. This quantity characterizes the steepness of the waves and the values $\epsilon = 0.1$ or 0.2 correspond to the weakly nonlinear regime. Therefore, for arbitrary given wave numbers it is easy to calculate the appropriate wave amplitudes.

Dependence on basin form

Now we are going to behavior difference in triad's behavior in different experimental basins. Suppose we a rectangular experimental basin; each specific mode has the form

$$[A_{mn}(T)\cos\frac{\omega t}{2} + B_{mn}(T)\sin\frac{\omega t}{2}]\cos\frac{m\pi x}{L_x}\cos\frac{n\pi y}{L_y}$$

where $A_{mn}(T), B_{mn}(T)$ are real and imaginary parts of slowly varying mode amplitudes; m and n are integers giving the number of half-lengths in the $x-$ and $y-$directions; L_x and L_y are the sides of the experimental basin and wave vector $\vec{k} = (k_x, k_y) = (2\pi m/L_x, 2\pi n/L_y)$. First notice that for the case $L_x = L_y = L$ the dispersion relation will take form

$$\omega^2(\vec{k}) = \frac{1}{L}gk + \frac{1}{L^3}\frac{\mu}{\nu}k^3$$

which means that taken above usual form of dispersion function corresponds to the unit square domain. The resonant solutions in a square with side L could be found as solution of resonance conditions with $\omega(\vec{k})$ written as a function of L. For instance, for $\mu/\nu = 16$ in square basin with side $L = 2$ there is a resonant triad

[1,14][23,13][24,27]; (25.0785, 50.2490, 75.3275).

Notice that this triad is not resonant in the square basin with $L = 1$ and vice versa, the triad

[1,6][4,5][5,11]; (15.5681, 16.2945, 31.8626)

is resonant for $L = 1$ and non-resonant for $L = 2$. Therefore, even in this very simple case when basin form does not change but only the basin sizes, no general selecting rule exists which allows us to find resonant triads in a given basin among the triads which are resonant in some other basin. Each time corresponding modification of dispersion function has to be used. For instance, for a rectangular basin the expression

$$\omega^2(\vec{k}) = g\frac{1}{L_xL_y}[(mL_y)^2+(nL_x)^2]^{1/2}+\frac{1}{(L_xL_y)^2}\frac{\mu}{\nu}[(mL_y)^2+(nL_x)^2]^{3/2}$$

has to be used.

Necessary experimental facilities

A few wave generators must be available to make it possible to generate a few different wave frequencies simultaneously. In order to measure vertical deformation of the water surface from its quiescent position a wave gauge and a video recorder can be used. A wide variety of liquids does provide observable results. An experimental basin must have walls movable in such a way that, for instance, the basin form remains rectangular while the ratio of the basin sides changes or, in case of a circular basin, only the radius of the basin changes.

4 Drift waves

Till now our main interest was to investigate resonantly interacting waves because they play significant role in the energy transfer *via* the wave spectrum. But the opposite opposite situation - behavior of *the waves which do not interact resonantly* could also be of a great interest in some physical problems, for instance, for drift waves in Tokamak. The following questions arise: How long do these waves keep their energy and does the confinement time change from one mode to another? Does there exist a way to characterize the confinement time for every specific wave? Does this time depend on the resonator form? etc. Evolution of drift waves in Tokamak plasma is described by Hasegawa-Mima equation which coincides up to renaming of variables with barotropic vorticity equation (BVE) which we used for numerical experiments described below. A possibility of energy concentration in a given set of drift waves due to some instability mechanisms in plasma was conjectured by V.I.Petviashvili. Basing on this conjecture, we performed numeric simulations in which the most

part of the total initial energy of the wave field (till 60%) was located within a few specific wave groups while the rest of the energy was distributed randomly (Gaussian) or equally among all other waves of the spectrum. The results of numerical simulations show that all the waves can be partitioned into three classes having qualitatively different energetic behavior.

Class A - active waves. It consists of the waves taking part in resonant interactions (RI) which are exact solutions of resonant conditions in general form

$$\omega(\vec{k}_1) \pm \omega(\vec{k}_2) \pm \omega(\vec{k}_3) = 0, \tag{4}$$

$$\vec{k}_1 \pm \vec{k}_2 \pm \vec{k}_3 = 0 \tag{5}$$

and of the near-resonant waves which can be defined as follows. Let us denote

$$R^1 = (S_1^1, S_2^1, S_3^1), R^2 = (S_1^2, S_2^2, S_3^2), ..., R^p = (S_1^p, S_2^p, S_3^p) \tag{6}$$

the set of all wave vectors giving exact solutions of Sys. (4),(5) in some finite spectral domain T so that T contains all these wave vectors and some others which we denote as T^r:

$$T = \{S_j^i\} \cup T^r \quad \forall i = 1, 2, ..p, \quad j = 1, 2, 3$$

while $r = 1, 2, 3, ..., (T^2 - 3p^{'})$ where $p^{'} \leq p$ depending on whether or not some of the triads R^j have common waves. Let us say that *waves are taking part in ARI* (approximate resonant interactions) if they have some non-zero frequency discrepancy

$$0 < \omega(\vec{k}_1) \pm \omega(\vec{k}_2) \pm \omega(\vec{k}_3) << 0 \tag{7}$$

while Eq.(5) is hold exactly. Let us consider all waves taking part in ARI and choose the triads containing *two waves* of some specific resonant triad R^j. Corresponding frequency discrepancy can be computed as

$$\Omega^j = \pm\omega(S_i^j) \pm \omega(S_k^j) \pm \omega(T^r). \tag{8}$$

Waves T_j^r taking part in these interactions are called *near-resonant waves.* The class of active waves A is now defined as follows:

$$A = \{S_j^i\} \cup T_l^r \subseteq T.$$

The wave $T^r_{l_0}$ at which the minimum of Ω^j is achieved for some specific j and $\forall T^r \in T$, is called a *minimal near–resonant wave* for the triad R^j. Energy leaves triad R^j *via* the wave $T^r_{l_0}$. Using iteratively this procedure of the construction of the minimal near-resonant wave for the triad (S^j_i, S^j_k, T^r) and so on, the way energy goes from one triad to another can be constructed. Energy oscillates first among the waves within resonant triad and then is re-distributed via a few near-resonant waves. The *active waves transfer energy through the spectrum.*

Class P - passive waves. It consists of the waves taking part in ARI, except near-resonant waves. Therefore frequency discrepancy could be found as $\Omega^j = \omega(P_i) \pm \omega(P_k) \pm \omega(P_r)$ where $P_j \in T \diagdown \{R^p\}$ and such p_0 does not exist for which some of waves $(P_i \& P_k), (P_i \& P_r), (P_k \& P_r)$ belong to resonant triad R^{p_0}. Passive waves conserve their energy during time scale corresponding to the three-wave interactions and longer. The confinement time in some specific passive waves depends on the initial energy distribution of the wave field. For the same initial conditions this time for different passive waves may vary 30-60% according to the magnitude of discrepancy.

Class N - neutral waves. It consists of the waves which do not belong to class 1 or class 2, i.e. both Eqs.(4),(5) are not satisfied, $N = T \backslash (A \cup P)$ and could be empty (see Table) depending on the form of resonator. But in case of $N \neq \varnothing$ the confinement time by these modes can be a few times longer than by passive modes. For illustration the graphs of the modes' energies are drawn with a relative shift by the axes Y.

Numeric studies of BVE show different energetic behavior of the modes belonging to different classes. More precisely, energy of the neutral mode, is conserved during the time $T_{conserve}$ much longer then those predicted by wave turbulence theory, sometimes 3 to 5 times longer. Energy of the passive modes, with $\Omega = 0.21$ and with $\Omega = 0.13$ conserves approximately during time $T_{conserve}/2$ and afterwards begin to spread over the wave spectrum. Energy of the active mode, $\Omega = 0$, is changing periodically from the very beginning. At the initial moment $T = 0$ all modes have the same amount of energy.

Therefore, for given dispersion relation (i.e. for given resonator form and boundary conditions) in the finite spectral domain it is possible to compute all its active, passive and neutral waves. The results for three different forms of resonators and two different spectral trun-

cations, T10 and T20, are following:

- **unit sphere**
  - T10: Number of active modes is 4 and number of neutral modes is 3;
  - T20: Number of active modes is 51 and number of neutral modes is 3;
- **square**
  - T10: Number of active modes is 15 and number of neutral modes is 0;
  - T20: Number of active modes is 53 and number of neutral modes is 0;
- **rectangular basin with side ratio 1/4**
  - T10: Number of active modes is 4 and number of neutral modes is 75;
  - T20: Number of active modes is 16 and number of neutral modes is 300.

Remark. The important fact is that the mode with given wave vector $\vec{k} = (m, n)$ may belong to different classes depending on the resonator geometry. For instance, the mode with wave vector $(2, 4)$ is active in a square and neutral in a rectangular basin with sides' ratio 1/4. Therefore, if we supply energy to the mode with a given frequency, the time of energy confinement by this mode is defined by the resonator geometry and we can easily increase the number of passive waves or decrease the number of active waves just by changing the sides' ratio of the basin. The resonator geometry is indeed crucial fact in all these considerations because the neutral modes do not exist in some geometries and therefore the best confinement time principally can not be obtained in them.

Our main hypothesis can now be formulated as follows

The H-mode discharge in Tokamaks could possibly be described in terms of three-waves interactions of the drift waves. The H-modes are the neutral modes or passive modes with big frequency discrepancy, the L-modes are the passive modes with small frequency discrepancy and ELM-modes are the active modes.

Let us construct a few parallels between the properties of H-mode discharge and the properties of the three-waves interactions in order

to support this hypothesis. We use the comprehensive review on H-mode discharges in Tokamaks [5] to extract the most important facts on the subject.

- The appearance of an H-mode discharge does not depend neither on the technique for its obtaining nor on the Tokamak geometry.

- The three-wave interaction dynamics is defined completely by the initial energy distribution among the modes and therefore does not depend on way of "putting" the energy into the specific waves. The fact of this dynamics existence also does not depend on the Tokamak geometry. The crucial fact is *the very existence* of resonator which means the boundary conditions importance; as soon as modes have lengths long enough "to notice" boundary conditions the dynamics described above will take place. *The specific geometry defines only which specific modes will be active, passive or neutral in this geometry.*

- The occurrence of repetitive instabilities at plasma edge - ELM-s. There exist ELM-s of the three types: "giant" (Type I), "grassy" (Type II) and Type III with big, small and intermediate frequencies correspondingly.

- The active waves perform the periodic energy exchange between the modes of the resonant or near-resonant triads. Numerical simulations show that the periodic energy oscillations within the modes of these triads take place and their frequencies belong to the three different types described above. We do not touch here the question about specifics of energetic behavior of the passive waves in the presence of some active waves (it is in itself very interesting and complicated problem) and only point out the fact that under some conditions the presence of active waves may decrease the confinement time of some specific passive waves.

- The H-mode is achieved only when a exceeded threshold in the heating power is exceeded but at the same time the improvement in core confinement is due to the reduction of turbulence.

- These high and low boundaries for the total wave field energy form, so to say, "the corridor" of weak nonlinearity: the treshold in the heating provides the transfer from the linear regime to a weakly nonlinear one while reduction of turbulence prevents transition to the strong nonlinear regime.

- The H-mode is initiated by the formation of the transport barrier; there exist a very strong temporal correlation between development of the transport barrier, a reduction in fluctuation of

amplitudes and a large change in the plasma edge; the mechanism by which the improvement in the core confinement is facilitated by the transport barrier is not known.

- The transport barrier formation leads to some specific profiles in the flow velocity and, therefore, to the specific energy distribution among the wave spectrum. Namely, the reduction of turbulence in the most part of the plasma core and increasing turbulence at the plasma periphery leads to the energy distribution needed for the resonant interactions become dominate, i.e. the most part of the wave field is settled into a few modes while the rest of energy is distributed more or less homogeneously among the others. If in an experiment we are able to increase the energy of some desired mode, we will obtain the theoretically predicted confinement.

The presented analysis is neither full nor exhaustive and in any case has only instructive importance because the dispersion relations under consideration are not realistic for toroidal plasma. Our purpose here is just to attract attention to the complicated problem of finding realistic dispersion function(s) for this very interesting physical problem. This hard common work of physicists and mathematicians may be well rewarded because as a result a new Tokamak geometry may evolve in which a lot of waves will improve their energy confinement time drastically.

5 Concluding remarks

We have shown that kinetic equation and clipping describe dynamics of two opposite limits of wave turbulence theory - continuous and discrete wave spectra correspondingly.

The natural - and highly non-trivial - problem to be studied now is to describe transition from one limit to another. One of the most tricky point here is following. As it was noticed in Section 3, discrete systems described by the same evolution PDE, the same eigenfunctions of linear part of original PDE and having functionally the same dispersion relation will have different resonant triads corresponding to slightly different boundary conditions so that triad ([1,6][4,5][5,11]) is resonant in square basin with a side $L = 1$ (in non-dimensional units) and is not resonant if $L = 2$.

Barotropic vorticity equation demonstrates much more complicated behavior, namely even the form of linear modes itself changes when regarded with different boundary conditions. For instance, BVE on a sphere has a linear wave

$$\psi_{sphere} = AP_n^m(\sin\phi)\exp i[m\lambda + \frac{2m}{n(n+1)}t], \tag{9}$$

where $P_n^m(x)$ is the associated Legendre function of degree n and order m; dispersion relation has then form

$$\omega_{sphere} = -2m/[n(n+1)]$$

and wave vector $\vec{k} = (m, n)$ has integer coordinates m, n. In the case of infinite plane BVE has a linear wave has form

$$\psi_{plane} = A\exp i(k_x x + k_y y + \omega) \tag{10}$$

and wave vector $\vec{k} = (k_x, k_y)$ has real coordinates k_x, k_y while dispersion relation is written out as

$$\omega_{plane} = k_x/(1 + k_x + k_y).$$

Thus, intuitively reasonable opinion, that it is enough to regard the resonance surface constructed for real variables and then find (if possible) its integer points, does not hold here. Transition from a sphere to a special infinite plane called β-plane can only be constructed locally, at some interaction latitude and this transition is possible **iff** this latitude exists [14]. In particularly, it means that 1) transition from a spherical domain to an infinite plane is in fact transition to a one-parametric family of infinite planes; and 2) such a transition is not always possible.

Nevertheless, these systems have much in common, not only general energetic behavior but also some more fine properties. For instance, "corridor" of weak nonlinearity described in previous Section and known from experimental plasma researches, has been established [1] in numerical computations with capillary waves. A very promising idea has been formulated in [1] - "to explain the slowdown of energy flux and lower Kolmogorov constant observed by Pushkarev and Zakharov in numerical experiments [13]" in terms of discrete system behavior. It is a highly demanding task indeed to prove this hypothesis because transition from kinetic equation to the original PDE with a given boundary conditions will be much more complicated than just changing the interaction domain within the frame of the same equation.

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